

$$4b) y' = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} y$$

$$(1-\lambda)(1-\lambda)(1-\lambda) - ((-3+3\lambda)+(-1+\lambda)) = (1-\lambda)^3 - (-4+\lambda) = (1-\lambda)^3 + 4 - 4\lambda$$

$$(1-\lambda)((1-\lambda)^2 + 4) = 0$$

$$\lambda_1 = 0$$

$$\lambda^3 - 2\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2}$$

$$\lambda_{1,2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm 2i$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} v_1 &= 0 \\ v_2 &= 1 \\ v_3 &= -1 \end{aligned}$$

$$u = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 0 \\ e^x \\ -e^x \end{bmatrix}$$

$$\lambda = 1+2i \quad \begin{bmatrix} 2i & -1 & -1 \\ 1 & 2i & 0 \\ 3 & 0 & 2i \end{bmatrix} \xrightarrow{(-3)} \sim \begin{bmatrix} 2i & -1 & -1 \\ 1 & 2i & 0 \\ 0 & -6i & 2i \end{bmatrix} \xrightarrow{(-2i)} \sim \begin{bmatrix} 2i & -1 & -1 \\ 0 & 3 & -1 \\ 0 & -6i & 2i \end{bmatrix} \xrightarrow{(-6i)} \sim \begin{bmatrix} 2i & -1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2i & -1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} 3x_2 - x_3 &= 0 \\ x_2 &= 1 \\ 3 - x_3 &= 0 \\ x_3 &= 3 \end{aligned}$$

$$x_1 2i - 1 - 3 = 0$$

$$x_1 2i = 4$$

$$x_1 = \frac{4}{2i} \cdot \frac{i}{i} = -2i$$

$$u_2 = \begin{bmatrix} -2i \\ 1 \\ 3 \end{bmatrix} \quad e^{(1+2i)x} \cdot \begin{bmatrix} -2i \\ 1 \\ 3 \end{bmatrix} = e^x (\cos 2x + i \sin 2x) \begin{bmatrix} -2i \\ 1 \\ 3 \end{bmatrix} = e^x \begin{bmatrix} -2i \cos 2x + 2 \sin 2x \\ \cos 2x + i \sin 2x \\ 3 \cos 2x + 3i \sin 2x \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 2e^x \sin 2x \\ e^x \cos 2x \\ 3e^x \cos 2x \end{bmatrix} \quad y_3 = \begin{bmatrix} -2e^x \cos 2x \\ e^x \sin 2x \\ 3e^x \sin 2x \end{bmatrix}$$

$$y = c_1 \begin{bmatrix} 0 \\ e^x \\ -e^x \end{bmatrix} + c_2 \begin{bmatrix} 2e^x \sin 2x \\ e^x \cos 2x \\ 3e^x \cos 2x \end{bmatrix} + c_3 \begin{bmatrix} -2e^x \cos 2x \\ e^x \sin 2x \\ 3e^x \sin 2x \end{bmatrix} \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$5) y_1' = -y_1 + y_2$$

$$y_2' = -y_2 + y_3$$

$$y_3' = y_1 - y_3$$

$$y_1(0) = 6$$

$$y_2(0) = -1$$

$$y_3(0) = 4$$

$$\begin{bmatrix} -1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 1 \\ 1 & 0 & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} & \lambda^3 + 2\lambda^2 + 1 \\ & (-1-\lambda)(-1-\lambda)(-1-\lambda) + 1 = (1+\lambda+\lambda^2)(-1-\lambda) + 1 = -\lambda^2 - 8\lambda - 4 - \lambda^3 - 2\lambda^2 - \lambda + 1 = \\ & = -\lambda^3 - 6\lambda^2 - 9\lambda \\ & \lambda^3 + 6\lambda^2 + 9\lambda = 0 \\ & \lambda(\lambda^2 + 6\lambda + 9) = 0 \end{aligned}$$

$$\lambda_1 = 0$$

$$\lambda_{2,3} = \frac{-6 \pm \sqrt{36-36}}{2}$$

$$\lambda = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$u = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} & \lambda_{2,3} = -3 \\ & v_2 + 4v_3 = 0 \\ & v_3 = 1 \\ & v_2 = -4 \\ & -v_1 + v_2 = 0 \\ & -v_1 - 4 = 0 \\ & v_1 = -4 \end{aligned}$$

$$\lambda_2 = -3$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} & v_2 + 2v_3 = 0 \\ & v_3 = 1 \\ & v_2 = -2 \\ & 2v_1 - 2 = 0 \\ & v_1 = 1 \end{aligned}$$

$$u = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \quad y = e^{-3x} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-3x} \\ -3e^{-3x} \\ e^{-3x} \end{bmatrix}$$

$$\lambda_3 = -3$$

$$\begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & 2 & 1 & | & -2 \\ 1 & 0 & -1 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 1 & 0 & -1 & | & 1 \\ 0 & 2 & 1 & | & -2 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & 1 & 2 & | & -1 \\ 0 & 2 & 1 & | & -2 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & 1 & 2 & | & -1 \\ 0 & 1 & 2 & | & -1 \end{bmatrix}$$

$$\begin{aligned} & v_2 + 2v_3 = -1 \\ & v_3 = 1 \\ & v_2 + 2 = -1 \\ & v_2 = -3 \end{aligned}$$

$$\begin{aligned} & 2v_1 - 3 = 1 \\ & v_1 = 2 \end{aligned}$$

$$v = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$y = e^{-3x} \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right) = e^{-3x} \begin{bmatrix} x+2 \\ -2x-3 \\ x+1 \end{bmatrix}$$

$$y = c_1 \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{-3x} \\ -3e^{-3x} \\ e^{-3x} \end{bmatrix} + c_3 e^{-3x} \begin{bmatrix} x+2 \\ -2x-3 \\ x+1 \end{bmatrix} \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$6 = -4c_1 + c_2 + c_3(0+2)$$

$$-1 = -4c_1 - 2c_2 + c_3(0-3)$$

$$4 = c_1 + c_2 + c_3$$

$$c_1 = 4 - c_2 - c_3$$

$$-4(4 - c_2 - c_3) + c_2 + 2c_3 = 6$$

$$-16 + 4c_2 + 4c_3 + c_2 + 2c_3 = 6$$

$$5c_2 + 6c_3 = 22$$

$$-4(4 - c_2 - c_3) - 2c_2 - 3c_3 = -1$$

$$-16 + 4c_2 + 4c_3 - 2c_2 - 3c_3 = -1$$

$$2c_2 + 6c_3 = 22$$

$$c_3 = 15 - 2c_2$$

$$5c_2 + 6(15 - 2c_2) = 22$$

$$5c_2 + 90 - 12c_2 = 22$$

$$c_2 = -\frac{68}{7}$$

$$c_3 = 15 - \frac{136}{7}$$

$$c_3 = -\frac{31}{7}$$

$$c_1 = 4 - \frac{68}{7} + \frac{31}{7}$$

$$c_1 = -\frac{9}{7}$$

$$y = -\frac{9}{7} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{68}{7} \begin{bmatrix} e^{-3x} \\ -3e^{-3x} \\ e^{-3x} \end{bmatrix} - \frac{31}{7} e^{-3x} \begin{bmatrix} x+2 \\ -2x-3 \\ x+1 \end{bmatrix}$$