RAK0018 ROMAN RAKUS PR. Č. 6 PAPÍR Č. 2

POKR. 2) $y' = -c_1 \sin x + c_2 \cos x - \left(\frac{2}{3}\sin^3 x \cos x\right)' - \left(\frac{2}{3}\cos^3 x \sin x\right)'$

$y' = -c_1 \sin x + c_2 \cos x - \frac{2}{3}\left((\sin^3 x)' \cos x + \sin^3 x(-\sin x)\right) - \frac{2}{3}\left((\cos^3 x)' \sin x + \cos^3 x \cos x\right) =$

$= -c_1 \sin x + c_2 \cos x - \frac{2}{3}\left(3\sin^2 x \cos^2 x - \sin^4 x\right) - \frac{2}{3}\left(3\cos^2 x(-\sin^2 x) + \cos^4 x\right) =$

$= -c_1 \sin x + c_2 \cos x - 2\sin^2 x \cos^2 x - \frac{2}{3}\sin^4 x + 2\cos^2 x \sin^2 x - \frac{2}{3}\cos^4 x =$

$= -c_1 \sin x + c_2 \cos x - \frac{2}{3}\left(\sin^4 x + \cos^4 x\right)$

$1 = -c_2 - \frac{2}{3}$

$c_2 = -\frac{5}{3}$

$y = -\cos x - \frac{5}{3}\sin x - \frac{2}{3}\sin^3 x \cos x - \frac{2}{3}\cos^3 x \sin x$

3) $y''' + y'' + y' + y = 0$

$\lambda^3 + \lambda^2 + \lambda + 1 = 0$

$\lambda^2(\lambda+1) + (\lambda+1) = 0$

$(\lambda+1)(\lambda^2+1) = 0$

$\lambda_1 = -1$

$\lambda_{2,3} = \pm i$

$y_1 = e^{-x}$

$y_{2,3} = e^{ix}$

$y_2 = \cos x$

$y_3 = \sin x$

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x \qquad c_1, c_2, c_3 \in \mathbb{R}$

4a) $y_1' = y_1 + y_2 + 1 + e^x$

$y_2' = 3y_1 - y_2$

$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

$(1-\lambda)(-1-\lambda) - 3 = 0$

$-1 - \lambda + \lambda + \lambda^2 - 3 = 0$

$\lambda^2 = 4$

$\lambda_{1,2} = \pm 2$

$\lambda_1 = 2 \quad \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \end{bmatrix}$

$-x_1 + x_2 = 0$

$x_1 = 1$

$x_2 = 1$

$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \quad \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \end{bmatrix}$

$3x_1 + x_2 = 0$

$x_1 = 1$

$x_2 = -3$

$u_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$y_1 = e^{2x}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^{2x}\\e^{2x}\end{bmatrix} \qquad y_2 = e^{-2x}\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}e^{-2x}\\-3e^{-2x}\end{bmatrix}$$

$$y_H = c_1\begin{bmatrix}e^{2x}\\e^{2x}\end{bmatrix} + c_2\begin{bmatrix}e^{-2x}\\-3e^{-2x}\end{bmatrix}$$

$$y_p = C_1(x)\begin{bmatrix}e^{2x}\\e^{2x}\end{bmatrix} + C_2(x)\begin{bmatrix}e^{-2x}\\-3e^{-2x}\end{bmatrix}$$

$$C_1'(x)\,e^{2x} + C_2'(x)\,e^{-2x} = 1+e^x$$

$$C_1'(x)\,e^{2x} + C_2'(x)(-3e^{-2x}) = 0$$

$$4C_2'(x)\,e^{-2x} = 1+e^x$$

$$4C_2'(x) = e^{2x}+e^{3x}$$

$$C_2'(x) = \frac{e^{2x}+e^{3x}}{4}$$

$$C_1'(x)\,e^{2x} + \frac{1}{4}(e^{2x}+e^{3x})\,e^{-2x} = 1+e^x$$

$$C_1'(x)\,e^{2x} + \frac{1}{4}e^x = 1+e^x$$

$$C_1'(x)\,e^{2x} = 1+e^x-\frac{1}{4}e^x$$

$$C_1'(x) = e^{-2x}+\frac{3}{4}e^{-x}$$

$$C_2(x) = \int\frac{1}{4}(e^{2x}+e^{3x})\,dx = \frac{1}{4}\left(\int e^{2x}dx+\int e^{3x}dx\right) = \left|\begin{matrix}s=2x\\ ds=2dx\end{matrix}\right| = \frac{1}{4}\left(\int e^s\frac{1}{2}ds+\int e^s\cdot\frac{1}{3}ds\right) =$$

$$= \frac{1}{4}\cdot\left(\frac{1}{2}e^{2x}+\frac{1}{3}e^{3x}\right) = \frac{1}{8}e^{2x}+\frac{1}{12}e^{3x}$$

$$C_1(x) = \int e^{-2x}+\frac{3}{4}e^{-x}\,dx = \int e^s\cdot\frac{1}{2}ds+\frac{3}{4}\int e^s\cdot(-1)\,ds = -\frac{1}{2}e^{-2x}-\frac{3}{4}e^{-x}$$

$$y_p = \left(-\frac{1}{2}e^{-2x}-\frac{3}{4}e^{-x}\right)\begin{bmatrix}e^{2x}\\e^{2x}\end{bmatrix} + \left(\frac{1}{8}e^{2x}+\frac{1}{12}e^{3x}\right)\begin{bmatrix}e^{-2x}\\-3e^{-2x}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}-\frac{3}{4}e^x\\-\frac{1}{2}-\frac{3}{4}e^x\end{bmatrix} + \begin{bmatrix}\frac{1}{8}+\frac{1}{12}e^x\\-\frac{3}{8}-\frac{1}{4}e^x\end{bmatrix} =$$

$$= \begin{bmatrix}-\frac{3}{8}-\frac{2}{3}e^x\\-\frac{7}{8}-e^x\end{bmatrix}$$

$$y = y_H + y_p = c_1\begin{bmatrix}e^{2x}\\e^{2x}\end{bmatrix} + c_2\begin{bmatrix}e^{-2x}\\-3e^{-2x}\end{bmatrix} + \begin{bmatrix}-\frac{3}{8}-\frac{2}{3}e^x\\-\frac{7}{8}-e^x\end{bmatrix} \qquad c_1, c_2 \in \mathbb{R}$$